Proceedings of the Jangjeon Mathematical Society
27 (2024), No. 2, pp. 205 – 217

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2024.27.2.205

On Graded ϕ-1-absorbing prime ideals

Mashhoor Refai, Rashid Abu-Dawwas, Ünsal Tekir, Suat Koç, Roa'a Awawdeh, and Eda Yildiz

Abstract. Let G be a group, R be a G-graded commutative ring with nonzero unity and $GI(R)$ be the set of all graded ideals of R. Suppose that $\phi : GI(R) \to GI(R) \cup \{\emptyset\}$ is a function. In this article, we introduce and study the concept of graded ϕ-1-absorbing prime ideals. A proper graded ideal I of R is called a graded ϕ-1-absorbing prime ideal of R if whenever a, b, c are homogeneous nonunit elements of R such that $abc \in I - \phi(I)$, then $ab \in I$ or $c \in I$. Several properties of graded ϕ-1-absorbing prime ideals have been examined.

1. Introduction

Throughout this article, G will be a group with identity e and R be a commutative ring having a nonzero unity 1. Then R is called a *G-graded ring* if $R = \bigoplus_{g\in G} R_g$ with $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$ where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are called *homogeneous of degree g*. If $a \in R$, then a can be written uniquely as a finite sum $\sum_{g\in G} a_g$, where a_g is the component of a in R_g. Note that R_e is a subring of R and $1 \in R_e$. The set of all homogeneous elements of R is denoted by $h(R) = \bigcup_{g\in G} R_g$. Let P be an ideal of a graded ring R. Then P is called a *graded ideal* if $P = \bigoplus_{g\in G}(P \cap R_g)$, or equivalently, $a = \sum_{g\in G} a_g \in P$ implies that $a_g \in P$ for all $g \in G$. It is not necessary that every ideal of a graded ring is a graded ideal. For instance, let $R = k[X]$ where k is a field. Then R is a $\mathbb{Z}$-graded ring where $R_n = 0$ if $n < 0$, $R_0 = k$ and $R_n = kX^n$ if $n > 0$. Then $I = (X + 1)$ is not a graded ideal since $1 + X \in I$ but $1 \notin I$. We will denote the set of all graded ideals of R by $GI(R)$. For more details and terminology, see [**8, 12**].

For many years, various classes of graded ideals have been established such as graded prime, graded primary, graded absorbing ideals, and etc. All of them play an important performance when characterizing graded rings. The concept of graded prime ideals and its generalizations have an important place in graded commutative algebra since they are used in recognizing the structure of graded rings. Recall that a proper graded ideal I of R is said to be a *graded prime ideal* if whenever $a, b \in h(R)$ such that $ab \in I$, then either $a \in I$ or $b \in I$ ([**14**]). The significance of graded prime ideals led many researchers to work on graded prime ideals and its generalizations. See for example, [**1, 3, 13**]. In [**7**], Atani introduced the notion of graded weakly prime ideal which is a generalization of graded prime ideals. A proper graded ideal I of R is said to be a *graded weakly prime ideal* of R if whenever $a, b \in h(R)$ such that $0 \neq ab \in I$, then $a \in I$ or $b \in I$. It is obvious that every graded prime ideal is

2010 *Mathematics Subject Classification.* Primary 13A02; Secondary 16W50.

Key words and phrases. Graded ϕ-prime ideal; graded 1-absorbing prime ideal; graded ϕ-1-absorbing prime ideal.

graded weakly prime but the converse is not true in general. For instance, consider the $\mathbb{Z}$-graded ring $R = \mathbb{Z}_4[X]$ and the ideal $I = (0)$. Then I is clearly a graded weakly prime ideal. However, I is not a graded prime ideal since $\overline{2} \cdot \overline{2}X = \overline{0}$ but $\overline{2}$ and $\overline{2}X \notin I$. Later, Al-Zoubi, Abu-Dawwas and Ceken in [**6**] introduced the notion of graded 2-absorbing ideals. A nonzero proper graded ideal I of R is called a *graded* 2-*absorbing ideal* if $abc \in I$ implies $ab \in I$ or $ac \in I$ or $bc \in I$ for each $a, b, c \in h(R)$. Note that every graded prime ideal is also a graded 2-absorbing ideal. After this, graded 2-absorbing version of graded ideals and many generalizations of graded 2-absorbing ideals attracted considerable attention by many researchers in [**2, 16, 18**]. In [**9**], the authors defined the notion of graded almost prime ideals. A proper graded ideal I of R is said to be *graded almost prime* if for $a, b \in h(R)$ such that $ab \in I - I^2$, then either $a \in I$ or $b \in I$. Also, in [**6**], the authors defined and studied graded *weakly* 2-*absorbing ideals* which is a generalization of graded weakly prime ideals. A proper graded ideal I of R is called a *graded weakly* 2-*absorbing ideal* if $0 \neq abc \in I$ implies $ab \in I$ or $ac \in I$ or $bc \in I$ for each $a, b, c \in h(R)$. In [**5**], Alshehry and Abu-Dawwas defined a new class of graded prime ideals. A proper graded ideal I of R is called *graded ϕ-prime ideal* if whenever $ab \in I - \phi(I)$ for some $a, b \in h(R)$, then either $a \in I$ or $b \in I$, where $\phi : GI(R) \to GI(R) \cup \{\emptyset\}$ is a function. They proved that a graded prime ideal and a graded ϕ-prime ideal have some similar properties.

Recently, in [**4**], the notion of graded 1-absorbing prime ideals has been introduced and studied. This class of graded ideals is a generalization of graded prime ideals. A proper graded ideal I of R is called a *graded* 1-*absorbing prime ideal* if whenever $abc \in I$ for some nonunits $a, b, c \in h(R)$, then either $ab \in I$ or $c \in I$. Note that every graded prime ideal is graded 1-absorbing prime and every graded 1-absorbing prime ideal is graded 2-absorbing ideal. The converses are not true. More currently, in [**17**], the notion of graded weakly 1-absorbing prime ideals which is a generalization of graded 1-absorbing prime ideals has been introduced and investigated. A proper graded ideal I of R is called a *graded weakly* 1-*absorbing prime ideal* if whenever $0 \neq abc \in I$ for some nonunits $a, b, c \in h(R)$, then either $ab \in I$ or $c \in I$.

In this article, we act in accordance with [**19**] to define and study graded ϕ-1-absorbing prime ideals as a new class of graded ideals which is a generalization of graded 1-absorbing prime ideals. A proper graded ideal I of R is called a *graded* ϕ-1-*absorbing prime ideal* of R if whenever $a, b, c \in h(R)$ are nonunits such that $abc \in I - \phi(I)$, then $ab \in I$ or $c \in I$. Among several results, an example of a graded weakly 1-absorbing prime ideal that is not graded 1-absorbing prime has been given (Example 6). Also, an example of a graded weakly 1-absorbing prime ideal that is not graded weakly prime has been introduced (Example 8). In Theorem 13, we give a characterization on graded ϕ-1-absorbing prime ideals. We introduce the concept of g-ϕ-1-absorbing prime ideals. A graded ideal I of R with $I_g \neq R_g$ is said to be a *g-ϕ-1-absorbing prime ideal of* R if whenever $a, b, c \in R_g$ such that $abc \in I$, then either $ab \in I$ or $c \in I$. In Theorem 16, we give a characterization of g-ϕ-1-absorbing prime ideals. We show that if I is a graded ϕ-1-absorbing prime ideal of R, then $I/\phi(I)$ is a graded weakly 1-absorbing prime ideal of $R/\phi(I)$ (Theorem 25). On the other hand, we prove that if $I/\phi(I)$ is a graded weakly 1-absorbing prime ideal of $R/\phi(I)$ and $U(R/\phi(I)) = \{a + \phi(I) : a \in U(R)\}$, then I is a graded ϕ-1-absorbing prime ideal of R (Theorem 27). In Theorem 28, we study graded ϕ-1-absorbing prime ideals over multiplicative sets. In Theorems 30, 31 and 32, we study graded ϕ-1-absorbing prime ideals over cartesian products of graded rings. Finally, we introduce and study the concept of graded von Neumann regular rings. A graded ring R is said to be a *graded von Neumann regular ring* if for each $a \in R_g$, there exists $x \in R_{g^{-1}}$ such that $a = a^2x$ [**12**]. In particular, we prove that if R is a graded von Neumann regular ring and $x \in h(R)$, then Rx is a graded almost 1-absorbing prime ideal of R (Theorem 40).

2. Graded ϕ-1-absorbing prime ideals

In this section, we introduce and study the concept of graded ϕ-1-absorbing prime ideals.

Definition 1. *Let R be a graded ring and $\phi : GI(R) \to GI(R) \cup \{\emptyset\}$ be a function. A proper graded ideal I of R is called a graded ϕ-1-absorbing prime ideal of R if whenever $a, b, c \in h(R)$ are nonunits such that $abc \in I - \phi(I)$, then $ab \in I$ or $c \in I$.*

Remark 2. *The following notations are used for the rest of the article, they are types of graded 1-absorbing prime ideals corresponding to ϕ_α.*

(1) $\phi_\emptyset(I) = \emptyset$ *(graded 1-absorbing prime ideal)*
(2) $\phi_0(I) = \{0\}$ *(graded weakly 1-absorbing prime ideal)*
(3) $\phi_1(I) = I$ *(any graded ideal)*
(4) $\phi_2(I) = I^2$ *(graded almost 1-absorbing prime ideal)*
(5) $\phi_n(I) = I^n$ *(graded n-almost 1-absorbing prime ideal)*
(6) $\phi_\omega(I) = \bigcap_{n=1}^{\infty} I^n$ *(graded ω-1-absorbing prime ideal)*

Remark 3. *(1) Since $I - \phi(I) = I - (I \bigcap \phi(I))$ for any graded ideal I, without loss of generality, throughout this article, we suppose that $\phi(I) \subseteq I$.*

(2) For functions $\phi, \psi : GI(R) \to GI(R) \cup \{\emptyset\}$, we write $\phi \leq \psi$ if $\phi(I) \subseteq \psi(I)$ for all $I \in GI(R)$. Obviously, therefore, we have the next order:

Remark 4. $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1$.

Proposition 5. *Let R be a graded ring, $\phi, \psi : GI(R) \to GI(R) \cup \{\emptyset\}$ be two functions with $\phi \leq \psi$ and I be a proper graded ideal of R.*

(1) *If I is a graded ϕ-1-absorbing prime ideal of R, then I is a graded ψ-1-absorbing prime ideal of R.*
(2) *I is a graded 1-absorbing prime ideal of $R \Rightarrow I$ is a graded weakly 1-absorbing prime ideal of $R \Rightarrow I$ is a graded ω-1-absorbing prime ideal of $R \Rightarrow I$ is a graded n-almost 1-absorbing prime ideal of R for each $n \geq 2 \Rightarrow I$ is a graded almost 1-absorbing prime ideal of R.*
(3) *I is a graded n-almost 1-absorbing prime ideal of R for each $n \geq 2$ if and only if I is a graded ω-1-absorbing prime ideal of R.*
(4) *Every graded ϕ-prime ideal of R is a graded ϕ-1-absorbing prime ideal of R.*

Proof. (1) : It is clear.

(2) : It follows from (1) and $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1$ in Remark 2.

(3) : By (2), if I is a graded ω-1-absorbing prime ideal of R, then I is a graded n-almost 1-absorbing prime ideal of R for each $n \geq 2$. Assume that I is a graded n-almost 1-absorbing prime ideal of R for each $n \geq 2$. Let $abc \in I - \bigcap_{n=1}^{\infty} I^n$ for some nonunits $a, b, c \in h(R)$. Then there exists $r \geq 2$ such that $abc \notin I^r$. Since I

is a graded r-almost 1-absorbing prime ideal of R and $abc \in I - I^r$, then either we have $ab \in I$ or $c \in I$.

(4) : It is obvious. □

The next example introduces a graded weakly 1-absorbing prime ideal that is not a graded 1-absorbing prime.

Example 6. *Consider $R = \mathbb{Z}_{pq^2}[i]$, where p, q are two distinct primes, and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}_{pq^2}$ and $R_1 = i\mathbb{Z}_{pq^2}$. As $\overline{q}^2 \in R_0$, $I = \langle \overline{q}^2 \rangle$ is a graded ideal of R. Since $\overline{p}, \overline{q} \in R_0 \subseteq h(R)$ are nonunits with $\overline{pqq} \in I$ while $\overline{pq} \notin I$ and $\overline{q} \notin I$, I is not a graded 1-absorbing prime ideal of R. On the other hand, we prove that I is a graded weakly 1-absorbing prime ideal of R. Let $\overline{0} \neq \overline{abc} \in I$ for some nonunits $\overline{a}, \overline{b}, \overline{c} \in h(R)$. Then q^2 divides abc but pq^2 does not divide abc.*

<u>Case (1):</u> $\overline{a}, \overline{b}, \overline{c} \in R_0$.

Since $\overline{a}, \overline{b}, \overline{c}$ are nonunits, p or q must divide a, b and c. If p divides a, b or c, then pq^2 divides abc which is a contradiction. So, q^2 divides ab and so $\overline{ab} \in I$. Therefore, I is a graded weakly 1-absorbing prime ideal of R.

<u>Case (2):</u> $\overline{a}, \overline{b} \in R_0, \overline{c} \in R_1$.

In this case, $\overline{c} = i\overline{\alpha}$ for some $\overline{\alpha} \in R_0$. As $\overline{c}$ is nonunit, $\overline{\alpha}$ is nonunit with $abc = iab\alpha$ and pq^2 does not divide $ab\alpha$. Since q^2 divides abc, $iab\alpha = q^2(x + iy)$ for some $x, y \in R_0$, and then $ab\alpha = q^2 y$ which implies that q^2 divides $ab\alpha$. Similarly as in case (1), we have that $\overline{ab} \in I$. Therefore, I is a graded weakly 1-absorbing prime ideal of R.

<u>Case (3):</u> $\overline{a} \in R_0$, $\overline{b}, \overline{c} \in R_1$.

In this case, $\overline{b} = i\overline{\alpha}$ and $\overline{c} = i\overline{\beta}$ for some $\overline{\alpha}, \overline{\beta} \in R_0$. As $\overline{b}$ and $\overline{c}$ are nonunits, $\overline{\alpha}$ and $\overline{\beta}$ are nonunits with $abc = -a\alpha\beta$ and pq^2 does not divide $a\alpha\beta$. Since q^2 divides abc, $-a\alpha\beta = q^2(x + iy)$ for some $x, y \in R_0$, and then $-a\alpha\beta = q^2 x$ which implies that q^2 divides $a\alpha\beta$. Similarly as in case (1), we have that $\overline{a\alpha} \in I$ and then $\overline{ab} \in I$. Therefore, I is a graded weakly 1-absorbing prime ideal of R.

<u>Case (4):</u> $\overline{a}, \overline{b}, \overline{c} \in R_1$.

In this case, $\overline{a} = i\overline{\alpha}$, $\overline{b} = i\overline{\beta}$ and $\overline{c} = i\overline{\gamma}$ for some $\overline{\alpha}, \overline{\beta}, \overline{\gamma} \in R_0$. As $\overline{a}$, $\overline{b}$ and $\overline{c}$ are nonunits, $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ are nonunits with $abc = -i\alpha\beta\gamma$ and pq^2 does not divide $\alpha\beta\gamma$. Since q^2 divides abc, $-i\alpha\beta\gamma = q^2(x + iy)$ for some $x, y \in R_0$, and then $-\alpha\beta\gamma = q^2 y$ which implies that q^2 divides $\alpha\beta\gamma$. Similarly as in case (1), we have that $\overline{\alpha\beta} \in I$ and then $\overline{ab} \in I$. Therefore, I is a graded weakly 1-absorbing prime ideal of R.

Since the other cases are similar to one of the above cases, I is a graded weakly 1-absorbing prime ideal of R.

The next example introduces a graded ω-1-absorbing prime ideal that is not graded weakly 1-absorbing prime.

Example 7. *Consider $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and the trivial graduation of R by any group G, that is $R_e = R$ and $R_g = \{0\}$ for $g \in G - \{e\}$. Now, $I = \mathbb{Z}_2 \times \{\overline{0}\} \times \{\overline{0}\} \times \{\overline{0}\}$ is a graded ideal of R satisfies $I^2 = I$, and then $I^n = I$ for all $n \geq 2$, and hence I is a graded ω-1-absorbing prime ideal of R. On the other hand, I is not a graded weakly 1-absorbing prime ideal of R since $a = (\overline{1}, \overline{1}, \overline{1}, \overline{0}), b = (\overline{1}, \overline{1}, \overline{0}, \overline{1})$ and $c = (\overline{1}, \overline{0}, \overline{1}, \overline{1}) \in h(R)$ are nonunits with $0 \neq abc \in I$ while $ab, c \notin I$.*

The next example introduces a graded weakly 1-absorbing prime ideal that is not graded weakly prime.

EXAMPLE 8. *Consider $R = \mathbb{Z}_{pq^2}[i]$, where p, q are two distinct primes, and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}_{pq^2}$ and $R_1 = i\mathbb{Z}_{pq^2}$. By Example 6, $I = \langle \overline{q}^2 \rangle$ is a graded weakly 1-absorbing prime ideal of R. On the other hand, I is not a graded weakly prime ideal of R since $\overline{q} \in h(R)$ with $\overline{0} \neq \overline{qq} \in I$ while $\overline{q} \notin I$.*

A graded ring R is said to be graded local if it has a unique graded maximal ideal $\mathfrak{m}$, and it is denoted by $(R, \mathfrak{m})$.

PROPOSITION 9. *Let $(R, \mathfrak{m})$ be a graded local ring and I be a proper graded ideal of R. If $\mathfrak{m}^2 \subseteq I$, then I is a graded 1-absorbing prime ideal of R.*

PROOF. Let $abc \in I$ for some nonunits $a, b, c \in h(R)$. Then $a, b, c \in \mathfrak{m}$, which implies that $ab \in \mathfrak{m}^2 \subseteq I$. Therefore, I is a graded 1-absorbing prime ideal of R. □

COROLLARY 10. *Let $(R, \mathfrak{m})$ be a graded local ring. Then $\mathfrak{m}^2$ is a graded 1-absorbing prime ideal of R.*

PROOF. By [[**4**], Lemma 1], $\mathfrak{m}^2$ is a proper graded ideal of R, and then $\mathfrak{m}^2$ is a graded 1-absorbing prime ideal of R by Proposition 9. □

PROPOSITION 11. *Let $(R, \mathfrak{m})$ be a graded local ring and I be a proper graded ideal of R. If $\mathfrak{m}^3 \subseteq \phi(I)$, then I is a graded ϕ-1-absorbing prime ideal of R.*

PROOF. Suppose that I is not a graded ϕ-1-absorbing prime ideal of R. Then there exist nonunit elements $a, b, c \in h(R)$ such that $abc \in I - \phi(I)$ but $ab \notin I$ and $c \notin I$. Since a, b, c are nonunits, they are elements of $\mathfrak{m}$, and then $abc \in \mathfrak{m}^3 \subseteq \phi(I)$, which is a contradiction. Hence, I is a graded ϕ-1-absorbing prime ideal of R. □

COROLLARY 12. *Let $(R, \mathfrak{m})$ be a graded local ring and $\phi(I) \neq \emptyset$ for every ideal I of R. If $\mathfrak{m}^3 = \{0\}$, then every proper graded ideal of R is graded ϕ-1-absorbing prime.*

PROOF. Apply Proposition 11. □

THEOREM 13. *Let R be a graded ring and I be a proper graded ideal of R. Consider the following conditions.*

(1) *I is a graded ϕ-1-absorbing prime ideal of R.*
(2) *For each nonunits $a, b \in h(R)$ with $ab \notin I$, $(I : ab) = I \cup (\phi(I) : ab)$.*
(3) *For each nonunits $a, b \in h(R)$ with $ab \notin I$, either $(I : ab) = I$ or $(I : ab) = (\phi(I) : ab)$.*
(4) *For each nonunits $a, b \in h(R)$ and proper graded ideal L of R such that $abL \subseteq I$ and $abL \nsubseteq \phi(I)$, either $ab \in I$ or $L \subseteq I$.*
(5) *For each nonunit $a \in h(R)$ and proper graded ideals K, L of R such that $aKL \subseteq I$ and $aKL \nsubseteq \phi(I)$, either $aK \subseteq I$ or $L \subseteq I$.*
(6) *For each proper graded ideals J, K, L of R such that $JKL \subseteq I$ and $JKL \nsubseteq \phi(I)$, either $JK \subseteq I$ or $L \subseteq I$.*

Then, $(6) \Rightarrow (5) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$.

PROOF. $(6) \Rightarrow (5)$: Suppose that $aKL \subseteq I$ and $aKL \nsubseteq \phi(I)$ for some nonunit $a \in h(R)$ and proper graded ideals K, L of R. Then $J = Ra$ is a graded ideal since $a \in h(R)$, and also $JKL \subseteq I$ and $JKL \nsubseteq \phi(I)$. Then by (6), we have $aK \subseteq JK \subseteq I$ or $L \subseteq I$ which completes the proof.

$(5) \Rightarrow (4)$: Let $abL \subseteq I$ and $abL \nsubseteq \phi(I)$ for some nonunits $a, b \in h(R)$ and proper graded ideal L of R. Now, put $K = Rb$. Then K is a graded ideal such that

$aKL \subseteq I$ and $aKL \nsubseteq \phi(I)$. Then by (5), we have that $ab \in aK \subseteq I$ or $L \subseteq I$ which is needed.

(4) $\Rightarrow$ (3) : Let $a, b \in h(R)$ nonunits such that $ab \notin I$. Then $(I : ab)$ is a proper graded ideal of R. We have two cases. **Case 1:** let $ab(I : ab) \subseteq \phi(I)$. Then $(I : ab) \subseteq (\phi(I) : ab)$. As the reverse inclusion always holds, we have the equality $(I : ab) = (\phi(I) : ab)$. **Case 2:** let $ab(I : ab) \nsubseteq \phi(I)$. Since $ab(I : ab) \subseteq I$, by (4), we get $(I : ab) \subseteq I$. As $I \subseteq (I : ab)$ always holds, we have $I = (I : ab)$. Therefore, $(I : ab) = I$ or $(I : ab) = (\phi(I) : ab)$.

(3) $\Rightarrow$ (2) : It is clear.

(2) $\Rightarrow$ (1) : Let $abc \in I - \phi(I)$ for some nonunits $a, b, c \in h(R)$. Assume that $ab \notin I$. Then we have $c \in (I : ab) - (\phi(I) : ab)$. By (2), we conclude that $c \in I$ which completes the proof. □

In the previous Theorem, the implication (1) $\nRightarrow$ (6) is not true in general. See the following example.

EXAMPLE 14. *Consider the ring $R = \mathbb{Z}_{50}[X]$. Then $R = \bigoplus_{n\in\mathbb{Z}} R_n$ is a $\mathbb{Z}$-graded ring, where $R_n = \{\overline{0}\}$ if $n < 0$, $R_0 = \mathbb{Z}_{50}$ and also $R_n = \mathbb{Z}_{50}X^n$ if $n > 0$. Then the set of all nonunit homogeneous elements is $nu(h(R)) = \{\overline{2k}, \overline{5k}, \overline{a}X^n : k, a \in \mathbb{Z}$ and $n \geq 1\}$. Now, consider the graded ideal $I = (X, \overline{25})$ of R. Set $\phi(I) = \{\overline{0}\}$. Now, we will show that I is a graded ϕ-1-absorbing prime ideal of R. To see this, choose nonunit homogeneous elements $r, s, t \in nu(h(R))$ such that $rst \in I - \phi(I)$. We have two cases. **Case 1:** If at least one of the r, s, t is of the form $\overline{a}X^n$, then we have $rs \in I$ or $t \in I$ since $X \in I$. **Case 2:** Assume that $r, s, t \in \{\overline{2k}, \overline{5k} : k \in \mathbb{Z}\}$. Then we can write $r = \overline{m}, s = \overline{n}, t = \overline{k}$ for some $m, n, k \in \mathbb{Z}$. Since $rst \in I - \phi(I)$, we have $25|mnk$ and $2 \nmid mnk$. Thus, 2 does not divide m, n and k. Which implies that $25|mn$ and so $rs \in I$. Therefore, I is a graded ϕ-1-absorbing prime ideal of R. Now, we will show that I does not satisfy (2) in Theorem 13. Now, take $a = \overline{2}$ and $b = \overline{5}$. Then note that $ab = \overline{10} \notin I$. Also, it is easy to see that $\overline{5}, X \in (I : ab)$. Then we have $\overline{5} + X \in (I : ab)$. On the other hand, note that $\overline{5} + X \notin (\phi(I) : ab) \cup I$. This shows that $(\phi(I) : ab) \cup I \subsetneq (I : ab)$. Thus, I does not satisfy (2), and so it does not satisfy all axioms (2) $-$ (6) in Theorem 13.*

DEFINITION 15. *Let R be a G-graded ring and $\phi : GI(R) \to GI(R) \cup \{\emptyset\}$ be a function. Suppose that $g \in G$ and I is graded ideal of R with $I_g \neq R_g$. Then I is called a g-ϕ-1-absorbing prime ideal of R if whenever $a, b, c \in R_g$ are nonunits such that $abc \in I - \phi(I)$, then $ab \in I$ or $c \in I$.*

THEOREM 16. *Let R be a G-graded ring, $g \in G$ and I be a graded ideal of R with $I_g \neq R_g$. Then the following statements are equivalent.*

(1) *I is a g-ϕ-1-absorbing prime ideal of R.*
(2) *For each nonunits $a, b \in R_g$ with $ab \notin I$, $(I :_{R_g} ab) \subseteq I \cup (\phi(I) :_{R_g} ab)$.*
(3) *For each nonunits $a, b \in R_g$ with $ab \notin I$, either $(I :_{R_g} ab) \subseteq I$ or $(I :_{R_g} ab) = (\phi(I) :_{R_g} ab)$.*
(4) *For each nonunits $a, b \in R_g$ and graded ideal J of R such that $J_g \neq R_g$, $abJ_g \subseteq I$ but $abJ_g \nsubseteq \phi(I)$, either $ab \in I$ or $J_g \subseteq I$.*
(5) *For each nonunit $a \in h(R)$ and graded ideals J, K of R such that $J_g \neq R_g$, $K_g \neq R_g$, $aJ_gK_g \subseteq I$ but $aJ_gK_g \nsubseteq \phi(I)$, either $aJ_g \subseteq I$ or $K_g \subseteq I$.*
(6) *For each graded ideals J, K, L of R such that $J_g \neq R_g$, $K_g \neq R_g$, $L_g \neq R_g$, $J_gK_gL_g \subseteq I$ but $J_gK_gL_g \nsubseteq \phi(I)$, either $J_gK_g \subseteq I$ or $L_g \subseteq I$.*

Proof. (1) $\Rightarrow$ (2) : Let $a, b \in R_g$ be nonunits with $ab \notin I$. Take $x \in (I :_{R_g} ab)$. Then we have $x \in R_g$ and $abx \in I$. Since $ab \notin I$, x is nonunit. As I is a g-ϕ-1-absorbing prime ideal of R, we conclude that $x \in I$ or $abx \in \phi(I)$. Which implies that $x \in I \cup (\phi(I) :_{R_g} ab)$. Thus, $(I :_{R_g} ab) \subseteq I \cup (\phi(I) :_{R_g} ab)$.

(2) $\Rightarrow$ (3) : Assume that $(I :_{R_g} ab) \subseteq I \cup (\phi(I) :_{R_g} ab)$. Then by [**11**], $(I :_{R_g} ab) \subseteq I$ or $(I :_{R_g} ab) \subseteq (\phi(I) :_{R_g} ab)$. In the first case, there is nothing to prove. Assume that $(I :_{R_g} ab) \subseteq (\phi(I) :_{R_g} ab)$. Since the reverse inclusion always holds, we have the equality $(I :_{R_g} ab) = (\phi(I) :_{R_g} ab)$.

(3) $\Rightarrow$ (4) : Suppose that $abJ_g \subseteq I$ but $abJ_g \nsubseteq \phi(I)$ for some nonunits $a, b \in R_g$ and graded ideal J of R with $J_g \neq R_g$. If $ab \in I$, then there is nothing to prove. So assume that $ab \notin I$. Since $J_g \subseteq (I :_{R_g} ab)$ and $J_g \nsubseteq (\phi(I) :_{R_g} ab)$, by (3), $J_g \subseteq (I :_{R_g} ab) \subseteq I$ which completes the proof.

(4) $\Rightarrow$ (5) : Suppose that $aJ_gK_g \subseteq I$ and $aJ_gK_g \subseteq \phi(I)$. Assume that $aJ_g \nsubseteq I$ and $K_g \nsubseteq I$. Then there exists $x \in J_g$ such that $ax \notin I$. Also, since $aJ_gK_g \nsubseteq \phi(I)$, there exists $y \in J_g$ such that $ayK_g \nsubseteq \phi(I)$. Now, assume that $axK_g \nsubseteq \phi(I)$. Since a, x are nonunits and $axK_g \subseteq I$, we have either $ax \in I$ or $K_g \subseteq I$, a contradiction. So, we get $axK_g \subseteq \phi(I)$. Also, we have $a(x + y)K_g \subseteq I$ and $a(x + y)K_g \nsubseteq \phi(I)$, which implies that $a(x + y) \in I$. Since $ayK_g \subseteq I$, $ayK_g \nsubseteq \phi(I)$ and $K_g \nsubseteq I$, we get $ay \in I$. Thus, we obtain $ax \in I$ giving a contradiction.

(5) $\Rightarrow$ (6) : Suppose that $J_gK_gL_g \subseteq I$ but $J_gK_gL_g \nsubseteq \phi(I)$ for some graded ideals J, K and L of R with $J_g \neq R_g$, $K_g \neq R_g$ and $L_g \neq R_g$. Assume that $J_gK_g \nsubseteq I$ and $L_g \nsubseteq I$. Then there exists $b \in J_g$ such that $bK_g \nsubseteq I$. Also, since $J_gK_gL_g \nsubseteq \phi(I)$, $aK_gL_g \nsubseteq \phi(I)$ for some $a \in J_g$. Then we get $aK_g \subseteq I$ since $aK_gL_g \subseteq I$ and $aK_gL_g \nsubseteq \phi(I)$. Suppose that $bK_gL_g \nsubseteq \phi(I)$. By (5), this gives $bK_g \subseteq I$ or $L_g \subseteq I$, which is a contradiction. So, $bK_gL_g \subseteq \phi(I)$. As $(a + b)K_gL_g \subseteq I$ and $(a + b)K_gL_g \nsubseteq \phi(I)$, we have $(a + b)K_g \subseteq I$. This implies $bK_g \subseteq I$, a contradiction.

(6) $\Rightarrow$ (1) : Let $abc \in I - \phi(I)$ for some nonunits $a, b, c \in R_g$. Then $(Ra)_g(Rb)_g(Rc)_g \subseteq I$ and $(Ra)_g(Rb)_g(Rc)_g \nsubseteq \phi(I)$. Hence, $(Ra)_g(Rb)_g \subseteq I$ or $(Rc)_g \subseteq I$ showing that $ab \in I$ or $c \in I$, as desired. □

Definition 17. *Let I be a g-ϕ-1-absorbing prime ideal of R and $a, b, c \in R_g$ be nonunits. Then (a, b, c) is called a g-ϕ-1-triple zero of I if $abc \in \phi(I)$, $ab \notin I$ and $c \notin I$.*

Theorem 18. *Suppose that I is a g-ϕ-1-absorbing prime ideal of R and (a, b, c) is a g-ϕ-1-triple zero of I. Then $abI_g \subseteq \phi(I)$.*

Proof. Now, $abc \in \phi(I)$, $ab \notin I$ and $c \notin I$. Suppose that $abI_g \nsubseteq \phi(I)$. Then there exists $x \in I_g$ such that $abx \notin \phi(I)$. So, $ab(c + x) \in I - \phi(I)$. If $c + x$ is unit, then $ab \in I$, a contradiction. Now, assume that $c+x$ is nonunit and so we get $ab \in I$ or $c \in I$, a contradiction. Thus, we have $abI_g \subseteq \phi(I)$. □

Theorem 19. *Suppose that I is a g-ϕ-1-absorbing prime ideal of R and (a, b, c) is a g-ϕ-1-triple zero of I. If $ac, bc \notin I$, then $acI_g \subseteq \phi(I)$, $bcI_g \subseteq \phi(I)$, $aI_g^2 \subseteq \phi(I)$, $bI_g^2 \subseteq \phi(I)$ and $cI_g^2 \subseteq \phi(I)$.*

Proof. Suppose that $acI_g \nsubseteq \phi(I)$. Then there exists $x \in I_g$ such that $acx \notin \phi(I)$. This implies that $a(b + x)c \in I - \phi(I)$. If $b + x$ is unit, then $ac \in I$ which is a contradiction. Thus $b + x$ is nonunit. Since I is a g-ϕ-1-absorbing prime ideal, we conclude either $a(b + x) \in I$ or $c \in I$, which implies that $ab \in I$ or $c \in I$, a contradiction. Thus, $acI_g \subseteq \phi(I)$. By using similar argument, we have $bcI_g \subseteq \phi(I)$.

Now, we will show that $aI_g^2 \subseteq \phi(I)$. Suppose not. Then there exist $x, y \in I_g$ such that $axy \notin \phi(I)$. It implies that $a(b+x)(c+y) \in I - \phi(I)$. If $(b+x)$ is unit, then $a(c+y) \in I$ which gives $ac \in I$, a contradiction. Similarly, $(c+y)$ is nonunit. Then either $a(b+x) \in I$ or $c+y \in I$ implying that $ab \in I$ or $c \in I$. Thus, we have $aI_g^2 \subseteq \phi(I)$. Similarly, we get $bI_g^2 \subseteq \phi(I)$ and $cI_g^2 \subseteq \phi(I)$. □

THEOREM 20. *Suppose that I is a g-ϕ-1-absorbing prime ideal of R and (a, b, c) is a g-ϕ-1-triple zero of I. If $ac, bc \notin I$, then $I_g^3 \subseteq \phi(I)$.*

PROOF. Suppose that $I_g^3 \nsubseteq \phi(I)$. Then there exist $x, y, z \in I_g$ such that $xyz \notin \phi(I)$, and then $(a+x)(b+y)(c+z) \in I - \phi(I)$. If $a+x$ is unit, then we obtain that $(b+y)(c+z) = bc+bz+cy+yz \in I$ and so $bc \in I$, which is a contradiction. Similarly, we can show that $b+y$ and $c+z$ are nonunits. Then we get $(a+x)(b+y) \in I$ or $c+z \in I$. This gives $ab \in I$ or $c \in I$, a contradiction. Hence, $I_g^3 \subseteq \phi(I)$. □

THEOREM 21. *Let R be a G-graded ring, $g \in G$ and $x \in R_g$ be nonunit. Suppose that $(0 : x) \subseteq Rx$. Then Rx is a g-ϕ-1-absorbing prime ideal of R with $\phi \leq \phi_2$ if and only if Rx is a g-1-absorbing prime ideal of R.*

PROOF. Suppose that Rx is a g-ϕ-1-absorbing prime ideal of R with $\phi \leq \phi_2$. Then it is also a g-ϕ_2-1-absorbing prime ideal of R by the sense of Proposition 5. Let $abc \in Rx$ for some nonunits $a, b, c \in R_g$. If $abc \notin (Rx)^2$, then $ab \in Rx$ or $c \in Rx$. Suppose that $abc \in (Rx)^2$. We have $ab(c+x) \in Rx$. If $c+x$ is unit, we are done. Hence, we can assume that $c+x$ is nonunit. Assume that $ab(c+x) \notin (Rx)^2$. Then we get either $ab \in Rx$ or $c+x \in Rx$ implying $ab \in Rx$ or $c \in Rx$. Now, assume that $ab(c+x) \in (Rx)^2$. This gives $xab \in (Rx)^2$ and so there exists $t \in R$ such that $xab = x^2t$. Thus we have $ab - xt \in (0 : x) \subseteq Rx$. Therefore, $ab \in Rx$, as needed. The converse is clear. □

REMARK 22. *Note that the condition $(0 : x) \subseteq Rx$ in Theorem 21 trivially holds for every regular element x.*

THEOREM 23. *Let R be a graded ring and I be a graded ideal of R with $I_e \neq R_e$. Suppose that R_e is not local ring and $(\phi(I) :_{R_e} a)$ is not maximal ideal of R_e for each $a \in I_e$. Then I is an e-ϕ-prime ideal of R if and only if I is an e-ϕ-1-absorbing prime ideal of R.*

PROOF. Suppose that I is an e-ϕ-1-absorbing prime ideal of R. Let $a, b \in R_e$ such that $ab \in I - \phi(I)$. If a or b is unit, then $a \in I$ or $b \in I$, as needed. Suppose that a, b are nonunits. Since $ab \notin \phi(I)$, $(\phi(I) :_{R_e} ab)$ is proper. Let $\mathfrak{m}$ be a maximal ideal of R_e containing $(\phi(I) :_{R_e} ab)$. Since R_e is not local ring, there exists another maximal ideal $\mathfrak{q}$ of R_e. Now, choose $c \in \mathfrak{q} - \mathfrak{m}$. Then $c \notin (\phi(I) :_{R_e} ab)$, and so we have $(ca)b \in I - \phi(I)$. Since I is an e-ϕ-1-absorbing prime ideal of R, we get either $ca \in I$ or $b \in I$. If $b \in I$, then we are done. Suppose that $ca \in I$. Then as $c \notin \mathfrak{m}$, there exists $x \in R_e$ such that $1 + xc \in \mathfrak{m}$. Note that $1 + xc$ is nonunit. If $1+xc \notin (\phi(I) :_{R_e} ab)$, then we have $(1+xc)ab \in I - \phi(I)$ implying $(1+xc)a \in I$ and so $a \in I$ since $ca \in I$. Assume that $1+xc \in (\phi(I) :_{R_e} ab)$, that is, $ab(1+xc) \in \phi(I)$. Choose $y \in \mathfrak{m} - (\phi(I) :_{R_e} ab)$. Then we have $(1 + xc + y)ab \in I - \phi(I)$. On the other hand, since $1 + xc + y \in \mathfrak{m}$, $1 + xc + y$ is nonunit. This implies that $(1 + xc + y)a \in I$. Also, since $yab \in I - \phi(I)$, we get $ya \in I$. Then we have $a = (1 + xc + y)a - x(ca) - ya \in I$. Therefore, I is an e-ϕ-prime ideal of R. The converse follows from Proposition 5. □

Let R be a G-graded ring and J be a graded ideal of R. Then R/J is a G-graded ring by $(R/J)_g = (R_g + J)/J$ for all $g \in G$. Moreover, we have the following:

Proposition 24. *([**15**], Lemma 3.2) Let R be a graded ring, J be a graded ideal of R and I be an ideal of R such that $J \subseteq I$. Then I is a graded ideal of R if and only if I/J is a graded ideal of R/J.*

For any graded ideal J of R define a function $\phi_J : GI(R/J) \to GI(R/J) \cup \{\emptyset\}$ by $\phi_J(I/J) = (\phi(I) + J)/J$ where $J \subseteq I$ and $\phi_J(I/J) = \emptyset$ if $\phi(I) = \emptyset$. Also, note that $\phi_J(I/J) \subseteq I/J$.

Theorem 25. *Let I be a graded ϕ-1-absorbing prime ideal of R. Then $I/\phi(I)$ is a graded weakly 1-absorbing prime ideal of $R/\phi(I)$.*

Proof. Let $0+\phi(I) \neq (a+\phi(I))(b+\phi(I))(c+\phi(I)) = abc+\phi(I) \in I/\phi(I)$ for some nonunits $a+\phi(I), b+\phi(I), c+\phi(I) \in R/\phi(I)$. Then a, b, c are nonunits in R and $abc \in I - \phi(I)$. Since I is a graded ϕ-1-absorbing prime ideal of R, $ab \in I$ or $c \in I$, and then we get $(a+\phi(I))(b+\phi(I)) = ab+\phi(I) \in I/\phi(I)$ or $c+\phi(I) \in I/\phi(I)$. Hence, $I/\phi(I)$ is a graded weakly 1-absorbing prime ideal of $R/\phi(I)$. □

Similarly, one can prove the following:

Theorem 26. *Let I, J be two graded ideals of R with $J \subseteq I$ and I be a graded ϕ-1-absorbing prime ideal of R. Then I/J is a graded ϕ_J-1-absorbing prime ideal of R/J.*

Theorem 27. *Let $I/\phi(I)$ be a graded weakly 1-absorbing prime ideal of $R/\phi(I)$ and $U(R/\phi(I)) = \{a+\phi(I) : a \in U(R)\}$. Then I is a graded ϕ-1-absorbing prime ideal of R.*

Proof. Let $a, b, c \in h(R)$ be nonunits such that $abc \in I - \phi(I)$. Then we have $0+\phi(I) \neq (a+\phi(I))(b+\phi(I))(c+\phi(I)) = abc+\phi(I) \in I/\phi(I)$. Since $U(R/\phi(I)) = \{a+\phi(I) : a \in U(R)\}$, $a+\phi(I), b+\phi(I), c+\phi(I)$ are nonunits in $R/\phi(I)$. Since $I/\phi(I)$ is a graded weakly 1-absorbing prime ideal, we have either $(a+\phi(I))(b+\phi(I)) = ab+\phi(I) \in I/\phi(I)$ or $c+\phi(I) \in I/\phi(I)$, which implies $ab \in I$ or $c \in I$. Therefore, I is a graded ϕ-1-absorbing prime ideal of R. □

Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicative set. Then $S^{-1}R$ is a G-graded ring with $(S^{-1}R)_g = \left\{\frac{a}{s} : a \in R_h, s \in S \cap R_{hg^{-1}}\right\}$ for all $g \in G$. If I is a graded ideal of R, then $S^{-1}I$ is a graded ideal of $S^{-1}R$. Consider the function $\phi : GI(R) \to GI(R) \cup \{\emptyset\}$. Define $\phi_S : GI(S^{-1}R) \to GI(S^{-1}R) \cup \{\emptyset\}$ by $\phi_S(S^{-1}I) = S^{-1}\phi(I)$ and $\phi_S(S^{-1}I) = \emptyset$ if $\phi(I) = \emptyset$. It is easy to see that $\phi_S(S^{-1}I) \subseteq S^{-1}I$.

Theorem 28. *Let R be a graded ring and $S \subseteq h(R)$ be a multiplicative set. If I is a graded ϕ-1-absorbing prime ideal of R with $I \cap S = \emptyset$, then $S^{-1}I$ is a graded ϕ_S-1-absorbing prime ideal of $S^{-1}R$.*

Proof. Let $\frac{a}{s}\frac{b}{t}\frac{c}{u} \in S^{-1}I - \phi_S(S^{-1}I)$ for some nonunits in $h(S^{-1}R)$. Then there exists $v \in S$ such that $vabc \in I$. If $vabc \in \phi(I)$, then we have $\frac{a}{s}\frac{b}{t}\frac{c}{u} = \frac{vabc}{vstu} \in S^{-1}\phi(I) = \phi_S(S^{-1}I)$ which is a contradiction. So we get $vabc \in I - \phi(I)$. Since va, b, c are nonunits in R and I is a graded ϕ-1-absorbing prime ideal, we get $vab \in I$ or $c \in I$. This implies $\frac{a}{s}\frac{b}{t} = \frac{vab}{vst} \in S^{-1}I$ or $\frac{c}{u} \in S^{-1}I$. Hence, $S^{-1}I$ is a graded ϕ_S-1-absorbing prime ideal of $S^{-1}R$. □

Let R and T be two G-graded rings. Then $R \times T$ is a G-graded ring by $(R \times T)_g = R_g \times T_g$ for all $g \in G$. Moreover, we have the following:

Proposition 29. *([**15**], Lemma 3.12) Let R and T be two graded rings. Then $L = I \times J$ is a graded ideal of $R \times T$ if and only if I is a graded ideal of R and J is a graded ideal of T.*

Let R and T be two graded rings, $\phi : GI(R) \to GI(R) \cup \{\emptyset\}$, $\psi : GI(T) \to GI(T) \cup \{\emptyset\}$ be two functions. Suppose that $\theta : GI(R \times T) \to GI(R \times T) \cup \{\emptyset\}$ is a function defined by $\theta(I \times J) = \phi(I) \times \psi(J)$ for each graded ideals I, J of R, T respectively. Then θ is denoted by $\theta = \phi \times \psi$.

Theorem 30. *Let R and T be two graded rings, $\phi : GI(R) \to GI(R) \cup \{\emptyset\}$, $\psi : GI(T) \to GI(T) \cup \{\emptyset\}$ be two functions. Suppose that $\theta = \phi \times \psi$. If $L = I \times J$ is a graded θ-1-absorbing prime ideal of $R \times T$, then I is a graded ϕ-prime ideal of R and J is a graded ψ-prime ideal of T.*

Proof. Let $a, b \in h(R)$ such that $ab \in I - \phi(I)$. Then we have $(a, 0)(1, 0)(b, 0) = (ab, 0) \in L - \theta(L)$ for some nonunits $(a, 0), (1, 0), (b, 0) \in h(R \times T)$. Since L is a graded θ-1-absorbing prime ideal of $R \times T$, we get either $(a, 0)(1, 0) = (a, 0) \in L$ or $(b, 0) \in L$ implying that $a \in I$ or $b \in I$. Therefore, I is a graded ϕ-prime ideal of R. Similarly, J is a graded ψ-prime ideal of T. □

Theorem 31. *Let R and T be two graded rings, $\phi : GI(R) \to GI(R) \cup \{\emptyset\}$, $\psi : GI(T) \to GI(T) \cup \{\emptyset\}$ be two functions. Suppose that $\theta = \phi \times \psi$. If $L = I \times J$ is a graded θ-1-absorbing prime ideal of $R \times T$ and $\theta(L_e) \neq L_e$, then $I = R$ or $J = T$.*

Proof. Since $\theta(L_e) \neq L_e$, either $\phi(I_e) \neq I_e$ or $\psi(J_e) \neq J_e$. Suppose that $\phi(I_e) \neq I_e$. Then there exists $a \in I_e - \phi(I_e)$ that is $a \in I - \phi(I)$. This implies that $(1, 0)(1, 0)(a, 1) = (a, 0) \in L - \theta(L)$. Then we have either $1 \in I$ or $1 \in J$, that is $I = R$ or $J = T$. Similarly, if $\psi(J_e) \neq J_e$, we have either $I = R$ or $J = T$. □

Theorem 32. *Let R and T be two graded rings, $\phi : GI(R) \to GI(R) \cup \{\emptyset\}$, $\psi : GI(T) \to GI(T) \cup \{\emptyset\}$ be two functions. Suppose that $\theta = \phi \times \psi$. Suppose that $L = I \times J$ is a graded θ-1-absorbing prime ideal of $R \times T$ and $\theta(L_e) \neq L_e$. If $\phi(R_e) \neq R_e$ is not a unique maximal ideal of R_e and $\psi(T_e) \neq T_e$ is not a unique maximal ideal of T_e, then either $L = R \times J$ and J_e is a prime ideal of T_e or $L = I \times T$ and I_e is a prime ideal of T_e*

Proof. By Theorem 31, we know that $I = R$ or $J = T$. Without loss of generality, we may assume that $I = R$. Let $xy \in J_e$ for some elemts $x, y \in T_e$. If x or y is unit, we are done. So assume that x, y are nonunits in T_e. Since $\phi(R_e) \neq R_e$ is not a unique maximal ideal of R_e, there exists a nonunit element $a \in R_e - \phi(R_e)$. Then we have $(a, 1)(1, x)(1, y) = (a, xy) \in L - \theta(L)$. Since I is a graded θ-1-absorbing prime ideal of $R \times T$, we have either $(a, 1)(1, x) = (a, x) \in L$ or $(1, y) \in L$ implying $x \in J$ or $y \in J$ that is either $x \in T_e \cap J = J_e$ or $y \in T_e \cap J = J_e$. Therefore, J_e is a prime ideal of T_e. □

3. Graded von Neumann regular Rings

In this section, we introduce and study the concept of graded von Neumann regular rings. We prove that if R is a graded von Neumann regular ring and $x \in h(R)$, then Rx is a graded almost 1-absorbing prime ideal of R (Theorem 40).

Definition 33. *Let R be a G-graded ring. Then R is said to be a graded von Neumann regular ring if for each $a \in R_g$ ($g \in G$), there exists $x \in R_{g^{-1}}$ such that $a = a^2x$.*

A graded commutative ring R with unity is said to be a graded field if every nonzero homogeneous element of R is unit [**15**]. Clearly, every field is a graded field, however, the converse is not true in general, see ([**15**], Example 3.6).

Lemma 34. *Let R be a graded ring. If $r \in R_g$ is a unit, then $r^{-1} \in R_{g^{-1}}$.*

Proof. By ([**12**], Proposition 1.1.1), $r^{-1} \in h(R)$, which means that $r^{-1} \in R_h$ for some $h \in G$. Now, $rr^{-1} = 1 \in R_e$ and $rr^{-1} \in R_gR_h \subseteq R_{gh}$. So, $0 \neq rr^{-1} \in R_e \cap R_{gh}$, which implies that $gh = e$, that is $h = g^{-1}$. Hence, $r^{-1} \in R_{g^{-1}}$. □

Example 35. *Every graded field is a graded von Neumann regular ring. To see this, let R be a graded field and $a \in R_g$. If $a = 0$, then $x = 0 \in R_{g^{-1}}$ satisfies $a = a^2x$. If $a \neq 0$, then a is unit, and then by Lemma 34, $x = a^{-1} \in R_{g^{-1}}$ with $a = a^2x$. Hence, R is a graded von Neumann regular ring.*

Lemma 36. *If R is a graded ring, then R_e contains all homogeneous idempotent elements of R.*

Proof. Let $x \in h(R)$ be an idempotent element. Then $x \in R_g$ for some $g \in G$ and $x^2 = x$. If $x = 0$, then $x \in R_e$ and we are done. Suppose that $x \neq 0$. Since $x^2 = x \cdot x \in R_gR_g \subseteq R_{g^2}$, $0 \neq x \in R_g \cap R_{g^2}$, and then $g^2 = g$ which implies that $g = e$, and hence $x \in R_e$. □

Proposition 37. *Let R be a graded ring. If R is a Boolean ring, then R is trivially graded.*

Proof. It is enough to prove that $R_g = \{0\}$ for all $g \neq e$. Let $g \in G - \{e\}$ and $x \in R_g$. Since R is Boolean, x is an idempotent, and then $x \in R_e$ by Lemma 36. So, $x \in R_g \cap R_e$ which implies the either $x = 0$ or $g = e$. Since $g \neq e$, $x = 0$, and hence R is trivially graded. □

Example 38. *Every Boolean graded ring is a graded von Neumann regular ring. To see this, let R be a Boolean graded ring. Then by Proposition 37, R is trivially graded. Assume that $a \in R_g$. If $g \neq e$, then $a = 0$ and then $x = 0 \in R_{g^{-1}}$ with $a = a^2x$. If $g = e$, then a is an idempotent, and then $x = a \in R_e = R_{g^{-1}}$ with $a^2x = ax = a \cdot a = a^2 = a$. Hence, R is a graded von Neumann regular ring.*

Lemma 39. *Let R be a graded von Neumann regular ring and $x \in h(R)$. Then $Rx = Ra$ for some idempotent element $a \in R_e$.*

Proof. Since $x \in h(R)$, $x \in R_g$ for some $g \in G$, and then there exists $y \in R_{g^{-1}}$ such that $x = x^2y$ as R is graded von Neumann regular. Choose $a = xy$, then $a = xy \in R_gR_{g^{-1}} \subseteq R_e$, and $a^2 = (xy) \cdot (xy) = (x^2y)y = xy = a$, which means that a is an idempotent. Now, $a = xy = yx \in Rx$, so $Ra \subseteq Rx$. On the other hand, $x = x^2y = x(xy) = xa \in Ra$, so $Rx \subseteq Ra$. Hence, $Rx = Ra$. □

Theorem 40. *Let R be a graded von Neumann regular ring and $x \in h(R)$. Then Rx is a graded almost 1-absorbing prime ideal of R.*

Proof. By [[**4**], Lemma 1], $I = Rx$ is a graded ideal of R. By Lemma 39, $I = Rx = Ra$ for some idempotent $a \in R_e$, and then $I^2 = I$ which implies that $I = Rx$ is a graded almost 1-absorbing prime ideal of R. □

Proposition 41. *Let R be a graded von Neumann regular ring and $x \in h(R)$. Then there exists an idempotent graded ideal J of R such that $R = Rx + J$ and $Rx \cap J = \{0\}$.*

Proof. By Lemma 39, $Rx = Ra$ for some an idempotent $a \in R_e$. Choose $J = R(1-a)$, then as $1-a \in R_e \subseteq h(R)$, J is a graded ideal of R by [[**4**], Lemma 1]. Also, $(1-a)^2 = 1-2a+a^2 = 1-2a+a = 1-a$ which means that $1-a$ is an idempotent, and so J is an idempotent ideal. Let $r \in R$. Then $r = ra + r(1-a) \in Ra + R(1-a) = Rx + J$, and hence $R = Rx + J$. Assume that $y \in Rx \cap J = Ra \cap J$. Then $y = \alpha a$ and $y = \beta(1-a)$ for some $\alpha, \beta \in R$. Now, $ya = \alpha a^2 = \alpha a = y$, and $ya = \beta(1-a)a = \beta a - \beta a^2 = \beta a - \beta a = 0$. So, $y = 0$, and hence $Rx \cap J = \{0\}$. □

Corollary 42. *If R is a graded von Neumann regular ring, then R is a direct sum of two idempotent graded ideals of R.*

Proof. Apply Proposition 41 and Lemma 39. □

References

[1] R. Abu-Dawwas, Graded semiprime and graded weakly semiprime ideals, Italian Journal of Pure and Applied Mathematics, 36 (2016), 535-542.
[2] R. Abu-Dawwas, M. Bataineh and H. Shashan, Graded generalized 2-absorbing submodules, Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry, (2020), DOI 10.1007/s13366-020-00544-1.
[3] M. Bataineh and R. Abu-Dawwas, On graded 2-prime ideals, Mathematics, (2021), https://doi.org/10.3390/math9050493.
[4] R. Abu-Dawwas, E. Yıldız, Ü. Tekir and S. Koç, On graded 1-absorbing prime ideals, Sao Paulo Journal of Mathematical Sciences, (2021), https://doi.org/10.1007/s40863-021-00218-3.
[5] A. S. Alshehry and R. Abu-Dawwas, On graded ϕ-prime submodules, arXiv:2102.04155, submitted.
[6] K. Al-Zoubi, R. Abu-Dawwas and S. Ceken, On graded 2-absorbing and graded weakly 2-absorbing ideals, Haceteppe Journal of Mathematics and Statistics, 48 (2019), 724-731.
[7] S. E. Atani, On graded weakly prime ideals, Turkish Journal of Mathematics, 30 (2006), 351-358.
[8] R. Hazrat, Graded rings and graded Grothendieck groups, Cambridge University press, 2016.
[9] A. Jaber, M. Bataineh and H. Khashan, Almost graded prime ideals, Journal of Mathematics and Statistics, 4 (4) (2008), 231-235.
[10] S. Koç, Ü. Tekir and E. Yıldız, On weakly 1-absorbing prime ideals, Ricerche mat (2021).
[11] N. H. McCoy, A note on finite unions of ideals and subgroups. Proceedings of the American Mathematical Society, 8(4) (1957), 633-637.
[12] C. Nastasescu and F. Oystaeyen, Methods of graded rings, Lecture Notes in Mathematics, 1836, Springer-Verlag, Berlin, 2004.
[13] M. Refai and K. Al-Zoubi, On graded primary ideals, Turkish Journal of Mathematics, 28 (2004), 217-229.
[14] M. Refai, M. Hailat and S. Obiedat, Graded radicals and graded prime spectra, Far East Journal of Mathematical Sciences, (2000), 59-73.
[15] H. Saber, T. Alraqad and R. Abu-Dawwas, On graded s-prime submodules, Aims Mathematics, 6 (2020), 2510-2524.
[16] F. Soheilnia and A. Y. Darani, On graded 2-absorbing and graded weakly 2-absorbing primary ideals, Kyungpook Mathematical Journal, 57 (2017), 559-580.
[17] Ü. Tekir, S. Koç, R. Abu-Dawwas and E. Yıldız, Graded weakly 1-absorbing prime ideals, submitted.
[18] R. N. Uregen, Ü. Tekir, K. P. Shum and S. Koç, On graded 2-absorbing quasi primary ideals, Southeast Asian Bulletin of Mathematics, 43 (4) (2019), 601-613.
[19] E. Yıldız, Ü. Tekir and S. Koç, On ϕ-1-absorbing prime ideals, Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry, (2021), https://doi.org/10.1007/s13366-020-00557-w.

President of Princess Sumaya University for Technology, Amman, Jordan
Email address: m.refai@psut.edu.jo

Department of Mathematics, Yarmouk University, Irbid, Jordan
Email address: rrashid@yu.edu.jo

Department of Mathematics, Marmara University, Istanbul, Turkey
Email address: utekir@marmara.edu.tr

Department of Mathematics, Marmara University, Istanbul, Turkey
Email address: suat.koc@marmara.edu.tr

Department of Mathematics, Yarmouk University, Irbid, Jordan
Email address: 2018105027@ses.yu.edu.jo

Department of Mathematics, Yildiz Technical University, Istanbul, Turkey
Email address: edyildiz@yildiz.edu.tr